

hep-th/9703039

PUPT-1684

UPT-739-T

Relativistic Brane Scattering

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Abstract

We calculate relativistic phase-shifts resulting from the large impact parameter scattering of 0-branes off p -branes within supergravity. Their full functional dependence on velocity agrees with that obtained by identifying the p -branes with D -branes in string theory. These processes are also described by 0-brane quantum mechanics, but only in the non-relativistic limit. We show that an improved 0-brane quantum mechanics based on a Born-Infeld type Lagrangian also does not yield the relativistic results. Scattering of 0-branes off bound states of arbitrary numbers of 0-branes and 2-branes is analyzed in detail, and we find agreement between supergravity and string theory at large distances to all orders in velocity. Our careful treatment of this system, which embodies the 11 dimensional kinematics of 2-branes in M(atric) theory, makes it evident that control of $1/n$ corrections will be necessary in order to understand our relativistic results within M(atric) theory.

^{*}Research supported in part by DOE grant DE-FG02-91-ER40671.

[†]Research supported in part by DOE grant AC02-76-ERO-3071.

1 Introduction

The realization that the Ramond-Ramond p -brane solutions to the $N = 2$ supergravities in ten dimensions have weak coupling descriptions in string theory as D -branes has substantially sharpened and extended the understanding of non-perturbative dualities relating string theories. The equivalence at large distances between p -branes and D -branes was originally established by considering static forces [1] and subsequently by scattering various probes off these objects [2, 3]. For example, the large impact parameter scattering of D -branes off D -branes has been analyzed in string theory by evaluating the cylinder exchange diagram [2, 4]. The resulting amplitudes are valid even for relativistic velocities, but the analogous processes in supergravity have only so far been calculated in the non-relativistic regime. In this paper we extend the classical results and show that, as expected, the equivalence between p -branes and D -branes continues to hold in relativistic processes. Although conceptually this agreement is a simple kinematical one, it nevertheless involves rather non-trivial functional dependences on the velocity of the probe.

D -branes also serve as probes that sense spacetime at distances smaller than the string scale [5, 6, 7, 8]. The identification of the new, shorter length scale as the Planck length of 11 dimensional supergravity is particularly interesting in view of the relationship of the string theories to 11 dimensional supergravity and its conjectured quantum description as M -theory [9, 10]. This has led to the proposal that M -theory in the infinite momentum frame is defined non-perturbatively by M(atr ix)-theory, identified as the $n \rightarrow \infty$ limit of the $SU(n)$ quantum mechanics describing a collection of n $D0$ -branes [11]. An important test of this proposal is the recovery of the supergravity scattering amplitudes at large impact parameter and small velocities from a description using only light open strings. However, as we shall show in detail, this agreement does not immediately extend to the complicated relativistic velocity dependence.

The disagreement we discuss is a direct consequence of the Galilean, rather than Lorentzian, invariance in the effective $SU(n)$ quantum mechanics. A suitable resummation of the open string theory might lead to a matching with supergravity. So we consider 0-brane scattering in the context of an improved Lagrangian - a supersymmetrized nonabelian Born-Infeld action that is manifestly relativistically invariant. However, we find that this simple remedy is not sufficient to recover the relativistic expressions for the brane scattering phase shifts.

If complete, M(atr ix)-theory should be able to reproduce the relativistic velocity dependences discussed in this paper. At leading order in velocity, M(atr ix)-theory relies on a high degree of supersymmetry as well as properties of light-cone kinematics to reproduce supergravity results at large distances. We will find that, even when these effects are taken into account, an apparent disagreement at sub-leading orders in velocity persists. In other words, the correct relativistic kinematics require either a substantial modification of the M(atr ix) Lagrangian or a highly non-trivial role of interactions. This conclusion is not surprising as relativistic invariance is known to be subtle in M(atr ix)-theory, if present at all. However, it is reached here from a new point of view that may be helpful for the understanding of M(atr ix)-theory dynamics.

The paper is organized as follows. In Sec. 2 we present the relativistic supergravity phase shifts for scattering of 0-branes off p -branes. In Sec. 3 we recall the full string theory results for the same processes, as well as their truncations to the lightest open and closed string modes. Finally, we discuss as yet unsuccessful attempts to reproduce our relativistic results from M(atr ix)-theory by employing, in Sec. 4, a supersymmetrized nonabelian Born-Infeld action and by carrying out, in Sec. 5, the highly boosted M(atr ix) kinematics.

2 Semiclassical Results

In this section we consider the large impact parameter scattering of a 0-brane probe off a p -brane target. The forces between branes arise from exchange of gravitons, dilatons, and, in the case of a 0-brane target, RR photons. We find the phase shifts for the scattering by lifting the process to 11 dimensions where it is purely gravitational: the probe 0-brane is a plane-fronted wave, and it simply follows a null-geodesic. This 11-dimensional interpretation of the scattering of branes greatly simplifies our calculations. We will completely neglect the backreaction of the probe 0-brane on the background geometry. The trivial kinematical backreaction could be accounted for by using the appropriate reduced mass but we will simply assume that the target is much heavier than the impinging probe. Another class of corrections corresponds to radiation emitted during collision. Such processes could be estimated semiclassically within the supergravity theory, but their reliable calculation is only possible in the full string theory. Fortunately radiative corrections can be neglected even at arbitrarily high impact velocity as long as the accelerations remain small. This is certainly the case when the impact parameter is large as it is in the situations that we study. Another interesting effect also beyond the semiclassical approximation is inelastic scattering exciting the internal structure of the target. This too is expected to be highly suppressed for large impact parameter scattering. With these limitations we shall find phase-shifts for 0-brane p -brane scattering that are valid to leading order in the impact parameter and to all orders in the probe velocity. The semi-classical phase shifts are derived from supergravity using the Hamilton-Jacobi formalism described in the next section.

2.1 The Hamilton-Jacobi method

The Hamilton-Jacobi functional S is a classical functional of particle path in the classical background. It is convenient to consider it as an ordinary function that depends on a single spacetime coordinate and a number of conserved quantities. The entire trajectory is defined by the requirement that it passes through the spacetime point, and the conserved quantities parametrize the space of such trajectories. The function S can be interpreted as the phase of a semiclassical wave function and is computed by solving the Hamilton-Jacobi equation:

$$g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} + m^2 = 0 . \quad (1)$$

This is simply the usual kinematical relation between momentum and mass and it is convenient to make this manifest by introducing

$$p_\alpha = \frac{\partial S}{\partial x^\alpha} . \quad (2)$$

The Hamilton-Jacobi equation can be solved explicitly with conserved quantities emerging as integration constants. A parametric form of the trajectory then follows by differentiation with respect to the conserved quantities [12, 13].

We wish to compute S , the phase of a semiclassical wave function for the scattering of 0-branes from p-brane backgrounds. A p-brane in 10 dimensions is described by the background [14, 15, 3]:

$$ds_{10}^2 = D_p^{-\frac{1}{2}}(-dt^2 + dx_1^2 + \cdots + dx_p^2) + D_p^{\frac{1}{2}}(dx_{p+1}^2 + \cdots + dx_9^2) \quad (3)$$

$$e^{-2\phi_{10}} = D_p^{\frac{p-3}{2}} \quad (4)$$

$$F_{p+2} = \partial_\mu D_p^{-1} dt \wedge dx_1 \wedge \cdots \wedge dx_p \wedge dx^\mu \quad (5)$$

$$D_p = 1 + \frac{q_p}{r^{7-p}} . \quad (6)$$

This solution can be lifted to 11 dimensions as:

$$ds_{11}^2 = e^{-\frac{2\phi}{3}} ds_{10}^2 + e^{\frac{4\phi}{3}} (dx_{11} - A_\mu dx^\mu)^2 \quad (7)$$

where A_μ is a Kaluza-Klein gauge field, interpreted as arising from 0-brane RR charge in 10 dimensions. A three-form field C is induced for $p = 2, 4$ but we will not write it explicitly because the 0-brane probe does not couple to it. A 0-brane scattering off a p-brane in 10 dimensions follows a null geodesic in the 11 dimensional background (Eq. 7) with some fixed momentum along the compact 11th dimension. The phase of the semiclassical wave function of the probe 0-brane is therefore given by solving the Hamilton-Jacobi equation (Eq. 1) with $m^2 = 0$.

2.2 0-2

As a first example of this formalism, we find the semiclassical phase shift for the scattering of 0-branes from 2-branes. The 2-brane background lifted to 11 dimensions is:

$$ds_{11}^2 = D_2^{-\frac{2}{3}}(-dt^2 + dx_1^2 + dx_2^2) + D_2^{\frac{1}{3}}(dx_3^2 + \cdots + dx_{11}^2) \quad (8)$$

The coordinates are labelled $(t, x_1, x_2, \cdots, x_9, x_{11})$ and $D_2 = 1 + \frac{q_2}{r^5}$ where $r^2 = x_3^2 + \cdots + x_9^2$. An elementary 2-brane in M -theory has identical form but with the harmonic function $D_2 = 1 + \frac{\tilde{q}_2}{r^6}$. The expression used here is recovered by averaging over the 11th dimension as appropriate for scattering at large impact parameters from a compactified brane. We consider only paths that do not depend on coordinates parallel to the brane, here x_1 and x_2 .

The classical trajectory of the probe 0-brane in 10 dimensions remains in one plane, due to angular momentum conservation. One angular variable named θ therefore

parametrizes the trajectory. In this coordinate system the Hamilton-Jacobi equation reads:

$$-D_2\left(\frac{\partial S}{\partial t}\right)^2 + \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \left(\frac{\partial S}{\partial x^{11}}\right)^2 = 0 \quad (9)$$

The equation can be solved by separation of variables. Conserved quantities

$$\frac{\partial S}{\partial t} = -E \quad (10)$$

$$\frac{\partial S}{\partial x^{11}} = p_{11} \quad (11)$$

$$\frac{\partial S}{\partial \theta} = J \quad (12)$$

appear as separation constants. It is then elementary to solve for $\frac{\partial S}{\partial r}$:

$$S = -Et + J\theta + p_{11}x^{11} + \int^r dr \sqrt{D_2E^2 - p_{11}^2 - \frac{J^2}{r^2}} \quad (13)$$

The classical trajectory is found in parametric form by differentiating with respect to E , J , and p_{11} :

$$t = \int^r dr \frac{D_2E}{\sqrt{D_2E^2 - p_{11}^2 - \frac{J^2}{r^2}}} \quad (14)$$

$$\theta = \int^r dr \frac{J}{r^2 \sqrt{D_2E^2 - p_{11}^2 - \frac{J^2}{r^2}}} \quad (15)$$

$$x^{11} = \int^r dr \frac{p_{11}}{\sqrt{D_2E^2 - p_{11}^2 - \frac{J^2}{r^2}}} \quad (16)$$

The indeterminate lower limits parametrize the arbitrary origin of each cyclic coordinate. We will not need these explicit expressions but it is of conceptual importance that the classical motion is present in the formalism.

A scattering phase shift, conventionally denoted 2δ , is obtained by subtracting from S of Eq. 13 the phase accumulated by a 0-brane with the same conserved charges moving in a flat background:

$$\delta_{02} = \int_{r_{\min}}^{\infty} dr \left[\sqrt{D_2E^2 - p_{11}^2 - \frac{J^2}{r^2}} - \sqrt{E^2 - p_{11}^2 - \frac{J^2}{r^2}} \right] \quad (17)$$

(The factor of 2 in 2δ was cancelled by the integral from ∞ to r_{\min} along the incoming part of the trajectory). As written, the lower integration limit r_{\min} in Eq. 17 is the classical turning point where the square root vanishes. The notation is formal because r_{\min} generically takes different values for the two terms which refer to motion in curved and flat backgrounds. However, we will only use Eq. 17 in the regime of the eikonal approximation where the energies are so large that the scattering angle remains small and r_{\min} can be taken to be the same for both terms.

2.3 0-4

Using the same techniques as in the previous section we can derive the phase shifts for 0-branes scattering off 4-branes. A 4-brane in 10 dimensions is lifted to 11 dimensions as:

$$ds_{11}^2 = D_4^{-\frac{1}{3}}(-dt^2 + dx_1^2 + \cdots + dx_4^2 + dx_{11}^2) + D_4^{\frac{2}{3}}(dx_5^2 + \cdots + dx_9^2) \quad (18)$$

where $D_4 = 1 + \frac{q_4}{r^3}$. This is a 5-brane in M -theory, averaged over one parallel dimension. The corresponding Hamilton-Jacobi equation is:

$$-D_4\left(\frac{\partial S}{\partial t}\right)^2 + \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + D_4\left(\frac{\partial S}{\partial x^{11}}\right)^2 = 0 \quad (19)$$

Conserved quantities are introduced as in Eqs. 10-12 and the Hamilton-Jacobi functional follows by quadrature, as before. The phase-shift becomes:

$$\delta_{04} = \int_{r_{\min}}^{\infty} dr \left[\sqrt{D_4(E^2 - p_{11}^2) - \frac{J^2}{r^2}} - \sqrt{E^2 - p_{11}^2 - \frac{J^2}{r^2}} \right] \quad (20)$$

2.4 0-0

The 0-brane lifts to:

$$ds_{11}^2 = -dt^2 + dx_1^2 + \cdots + dx_{11}^2 + (D_0 - 1)(dt + dx_{11})^2 \quad (21)$$

where $D_0 = 1 + \frac{q_0}{r^7}$. This metric represents a gravitational wave in 11 dimensions, averaged over its longitudinal direction. Alternatively it can be interpreted as a Schwarzschild black hole in 10 dimensions boosted by an infinite amount with the black hole mass taken to zero as the boost parameter is taken to infinity. The Hamilton-Jacobi equation for another 0-brane propagating in this background becomes:

$$-\left(\frac{\partial S}{\partial t}\right)^2 + \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \left(\frac{\partial S}{\partial x^{11}}\right)^2 - (D_0 - 1)\left(\frac{\partial S}{\partial t} - \frac{\partial S}{\partial x^{11}}\right)^2 = 0 \quad (22)$$

Again the conserved quantities are Eqs. 10-12. Note that in this case, unlike the previous ones, the relative sign of E and p_{11} matters. Identical signs give brane-brane ($0 - 0$) scattering and opposite ones give brane-anti-brane ($0 - \bar{0}$) scattering. The phase-shift becomes:

$$\delta_{00} = \int_{r_{\min}}^{\infty} dr \left[\sqrt{E^2 - p_{11}^2 - \frac{J^2}{r^2} + \frac{q_0}{r^7}(E \mp p_{11})^2} - \sqrt{E^2 - p_{11}^2 - \frac{J^2}{r^2}} \right] \quad (23)$$

Here the upper (lower) sign corresponds to $0 - 0$ ($0 - \bar{0}$).

2.5 0-6

The manifestation in 11 dimensions of a 6-brane in 10 dimensions is the Kaluza-Klein monopole:

$$ds_{11}^2 = (-dt^2 + dx_1^2 + \cdots + dx_6^2) + D_6(dx_7^2 + dx_8^2 + dx_9^2) + D_6^{-1}(dx_{11} + q_6(1 - \cos \theta)d\phi)^2 \quad (24)$$

where $D_6 = 1 + \frac{q_6}{r}$. In this case the scattering is three dimensional, as we shall explain below. We employ spherical coordinates with $\theta = 0$ in the initial state and we would have $\theta = \pi$ in the final state if there were no scattering. The azimuthal angle ϕ is $\phi = 0$ in the initial state. The Hamilton-Jacobi equation is:

$$-\left(\frac{\partial S}{\partial t}\right)^2 + D_6^{-1}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi} - q_6(1 - \cos \theta)\frac{\partial S}{\partial x^{11}}\right)^2\right] + D_6\left(\frac{\partial S}{\partial x^{11}}\right)^2 = 0 \quad (25)$$

Here θ is not a cyclic coordinate so the obvious separation constants eqs. 10–11 and

$$\frac{\partial S}{\partial \phi} = p_\phi \quad (26)$$

must be supplemented with

$$J^2 = \left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{\sin^2 \theta}(p_\phi - q_6 p_{11}(1 - \cos \theta))^2 \quad (27)$$

instead of eq. 12. The Hamilton-Jacobi functional reads:

$$S = -Et + p_\phi \phi + p_{11} x^{11} + \int^\theta d\theta \sqrt{J^2 - \frac{1}{\sin^2 \theta}(p_\phi - q_6 p_{11}(1 - \cos \theta))^2} + \int^r dr \sqrt{D_6 E^2 - D_6^2 p_{11}^2 - \frac{J^2}{r^2}} \quad (28)$$

The 0–6 scattering is more complicated than previous cases because the orientation of the orbital angular momentum is not conserved. However, its magnitude is conserved and the total angular momentum vector is maintained by the electro-magnetic field responsible for the interaction between the monopole and the charge. There is still no loss of generality in choosing coordinates such that $p_\phi = 0$ and it is convenient to do so. However, taking derivatives with respect to p_ϕ before setting it to zero, we see that the trajectory in ϕ coordinates is non-trivial and the motion accordingly non-planar, as expected when the direction of orbital angular momentum varies in time. Note that p_ϕ is the canonical momentum rather than the kinematical one; so it can vanish even when there is motion in the ϕ direction. The final state has $\phi_f = \frac{\pi}{2}$ and $\cot \frac{\theta_f}{2} = \frac{q_6 p_{11}}{J}$. This can be shown explicitly by carrying out the relevant integrals. (Remarkably, this also follows using nothing but angular momentum conservation.) These details are not relevant for what follows and mentioned only for completeness.

The phase-shift is given as:

$$\delta_{06} = \frac{1}{2} \int_0^{\theta_f} d\theta \left[\sqrt{J^2 - q_6^2 p_{11}^2 \tan^2 \frac{\theta}{2}} - J \right] + \int_{r_{\min}}^\infty dr \left[\sqrt{D_6 E^2 - D_6^2 p_{11}^2 - \frac{J^2}{r^2}} - \sqrt{E^2 - p_{11}^2 - \frac{J^2}{r^2}} \right] \quad (29)$$

2.6 The Eikonal approximation

The conserved quantities are related to initial conditions through:

$$E = \mu_0 \frac{1}{\sqrt{1-v^2}} \quad (30)$$

$$p_{11} = \mu_0 \quad (31)$$

$$p_{10} = \mu_0 \frac{v}{\sqrt{1-v^2}} \quad (32)$$

$$\frac{J}{p_{10}} = b \quad (33)$$

where μ_o is the probe 0-brane mass, v is the probe velocity, and b is the impact parameter. For convenience we also introduced the auxiliary quantity p_{10} , the magnitude of 10-dimensional momentum, defined through $E^2 = p_{10}^2 + p_{11}^2$. Eq. 33 equates the angular momentum denoted J with the magnitude of the orbital angular momentum $\vec{r} \times \vec{p}$. In the $0-6$ case a careful distinction was made between particle angular momentum and total angular momentum, but conventions were chosen so that Eq. 33 remains valid in this case.

The eikonal approximation is valid when the interaction is a small perturbation in the sense that the two square roots in the expressions for the phase shifts are comparable. Then the first square root can be expanded and the lower limit can be taken at $r_{\min} = b$. For example the phase shift Eq. 17 becomes

$$\delta_{02} \simeq \frac{E^2}{2} \int_b^\infty dr \frac{q_2}{r^5} \frac{1}{\sqrt{p_{10}^2 - \frac{J^2}{r^2}}} \quad (34)$$

$$= \frac{q_2 \mu_0}{2v \sqrt{1-v^2}} \frac{1}{b^4} I_2 \quad (35)$$

where

$$I_p = \int_0^\infty dx \frac{1}{(x^2 + 1)^{\frac{7-p}{2}}} = \frac{1}{2} \sqrt{\pi} \frac{\Gamma(\frac{6-p}{2})}{\Gamma(\frac{7-p}{2})}. \quad (36)$$

The substitution $z^2 = r^2 - \frac{J^2}{p_{10}^2}$ is helpful in the intermediate step. Analogous calculations in the other cases yield

$$\delta_{00} \simeq \frac{1}{2} q_0 \mu_0 \frac{(1 \mp \sqrt{1-v^2})^2}{v \sqrt{1-v^2}} \frac{1}{b^6} I_0 \quad (BB/BA) \quad (37)$$

$$\delta_{04} \simeq \frac{1}{2} q_4 \mu_0 \frac{v}{\sqrt{1-v^2}} \frac{1}{b^2} I_4 \quad (38)$$

$$\delta_{06} \simeq \frac{1}{2} q_6 \mu_0 \frac{2v^2 - 1}{v \sqrt{1-v^2}} I_6 \quad (39)$$

In the case of $0-6$ the complications due to angular momentum exchange are of higher order in q_6 and are not captured by the eikonal approximation. Note that δ_{06} is in fact divergent because I_6 is logarithmically divergent. The formal expression for

the phase shift is nevertheless of interest as it will also emerge (with the same caveats) from string theory. With our conventions a positive phase shift signals an attractive force. Only the 6-brane is repulsive, and then only for small v . Note however that the concept of a force, and that of a potential, is quite subtle in this context because of the velocity dependent nature of the interaction. The phase shifts are unambiguous, of course, and a potential can be formally defined from them.

It is instructive to compare the validity of the eikonal approximation for different values of p . The requirement is that we should be able to expand the square root in the phase shift. Using the fact that $(p_{10}^2 - J^2/r^2)$ is $O(p_{10}^2)$ throughout the important region of the phase shift integral we can easily show that the eikonal expansion can be made when¹:

$$\frac{q_p}{b^{7-p}} \ll v^\alpha \quad (40)$$

Here $\alpha = 2$ for $(0 - \bar{0}, 0 - 2, 0 - 6)$, $\alpha = 0$ in the $0 - 4$ case, and $\alpha = -2$ for $0 - 0$. In the fully relativistic regime where $v \sim 1$ the distinctions between the various cases disappear. For very small velocities we see that, for $0 - 0$, the eikonal approximation remains valid even for impact parameters much smaller than the scale set by q_p . In string theory this raises the interesting possibility of a simple approximation scheme, valid well below the string scale. The precise conditions obtained in Eq. 40 from kinematical reasoning agree with those observed in [8]. There they were understood from non-renormalization theorems by noting that, in the limit of vanishing velocity, the different cases $p = \bar{0}, 2, 6$, $p = 4$, and $p = 0$ preserve none, $\frac{1}{4}$, and $\frac{1}{2}$ of the supersymmetry, respectively.

2.7 Charge quantization

In the string theoretic calculation (reviewed in Sec. 3) the velocity dependence of phase shifts is parametrized by the function:

$$F_p(v) = \frac{(8 - 2p) + 4 \sinh^2 \pi \epsilon \mp \delta_{p,0} 8 \cosh \pi \epsilon}{4 \sinh \pi \epsilon} \quad (41)$$

where $\cosh \pi \epsilon = \frac{1}{\sqrt{1-v^2}}$. The supergravity phase shifts can also be written in terms of $F_p(v)$ as:

$$\delta_{0p} = \frac{1}{2} \frac{q_p \mu_0}{b^{6-p}} I_p F_p(v) \quad (42)$$

Recall that q_p are charges that appear as coefficients in the harmonic functions $D_p = 1 + \frac{q_p}{r^{7-p}}$. As shown in Appendix A using purely semiclassical reasoning, the q_p are related to the quantized charge *i.e.* the number of branes n_p , through:

$$q_p = \frac{l_s^{7-p} g}{2\pi \omega_{6-p}} n_p \quad (43)$$

¹A careful examination is required to reach this conclusion because $p_{10}^2 - J^2/r^2 = 0$ at the turning point. However, it is sufficient that $p_{10}^2 - J^2/r^2$ remains of $O(p_{10}^2)$ in the bulk of the important region of the integral.

The string length is $l_s = 2\pi\sqrt{\alpha'}$ and the volume of S_{6-p} is:

$$\omega_{6-p} = \frac{2\pi^{\frac{7-p}{2}}}{\Gamma(\frac{7-p}{2})} \quad (44)$$

We also need the 0-brane mass $\mu_0 = \frac{2\pi}{l_{sg}}$. In terms of these quantities, the supergravity phase shifts are:

$$2\delta_{0p} = \frac{\Gamma(\frac{6-p}{2})}{4\pi^{\frac{6-p}{2}}} \left(\frac{l_s}{b}\right)^{6-p} n_p F_p(v) . \quad (45)$$

In this form the supergravity phase shifts can be compared with results from string theory.

3 String Theory

The $0 - p$ brane scattering phase-shifts have also been calculated in string theory to the leading order in the genus expansion [2, 4]. The result is:

$$2\delta_{0p} = \int_0^\infty \frac{dt}{2\pi t} e^{-\frac{tb^2}{2\pi\alpha'}} B_p J_p \quad (46)$$

where

$$\begin{aligned} B_p &= f_1^{-(8-p)}(q) f_4^{-p}(q) \frac{\Theta'_1(0|it)}{\Theta_1(\epsilon t|it)} \\ J_p &= \frac{1}{2} [-f_2^{8-p}(q) f_3^p(q) \frac{\Theta_2(\epsilon t|it)}{\Theta_2(0|it)} + f_3^{8-p}(q) f_2^p(q) \frac{\Theta_3(\epsilon t|it)}{\Theta_3(0|it)} \mp \delta_{p,0} f_4^8(q) \frac{\Theta_4(\epsilon t|it)}{\Theta_4(0|it)}] \end{aligned} \quad (47)$$

The functions f_i and Θ_i are modular functions in the standard notation [16, 17] and $q = e^{-\pi t}$.

3.1 Closed String Expansion

At large distances the amplitudes are dominated by the exchange of massless closed string states with massive states damped at scales of order α' . This regime is dominated by small values of t in the integral Eq. 46. The behavior of the integrand for small t is found by a modular transformation followed by expansion in $e^{-\frac{\pi}{t}}$. The result is:

$$B_p J_p \simeq 2^{-\frac{p-4}{2}} \pi t^{\frac{6-p}{2}} F_p(v) \quad (48)$$

where F_p was introduced in Eq. 41. The phase-shift should also be multiplied with n_p , the number of target branes. Inserting Eq. 48 in Eq. 46, and applying the formula

$$\int_0^\infty dt t^{\frac{4-p}{2}} e^{-\frac{tb^2}{2\pi\alpha'}} = \Gamma\left(\frac{6-p}{2}\right) \left(\frac{2\pi\alpha'}{b^2}\right)^{\frac{6-p}{2}} \quad (49)$$

the phase-shift becomes:

$$2\delta_{0p} = \frac{1}{4\pi^{\frac{6-p}{2}}} \Gamma\left(\frac{6-p}{2}\right) \left(\frac{l_s}{b}\right)^{6-p} n_p F_p(v) \quad (50)$$

This is the same as the supergravity expression Eq. 45. It has not been checked before that this agreement indeed holds to all orders in v as it should.

3.2 Open String Expansion

At distances much smaller than the string length the interaction is dominated by open strings stretching between the target and the probe. The appropriate terms are isolated by expanding the integrand of Eq. 46 for large t . We only know the string theory amplitudes in the eikonal approximation and for $p = \bar{0}, 2, 6$ this is only valid at distances much larger than the string length; so there is no regime dominated by open strings and also captured by the eikonal approximation. Accordingly, for these cases the integration giving the phase shift Eq. 46 diverges at small t when the large t approximations to the kernels are inserted. On the other hand, for $p = 0$ and v very small, the eikonal approximation is valid at much smaller distances and there is a regime where open strings dominate in a controlled way. In fact, this is also true for $p = 4$ [8]. Recalling that $\pi\epsilon \simeq v$ for small v and expanding the integrand of Eq. 46 for large t gives:

$$B_0 J_0 \simeq \pi \frac{12 + 4 \cos 2vt - 16 \cos vt}{2 \sin vt} = 4\pi \frac{(1 - \cos vt)^2}{\sin vt} \quad (51)$$

$$B_4 J_4 \simeq 2\pi \frac{1 - \cos vt}{\sin vt} \quad (52)$$

Inserting these expressions into Eq. 46, the open string approximation to the phase shift is obtained. A surprising feature is noticed directly from Eqs. 51-52: expanding to the leading order in v , the closed string approximation Eq. 48 is recovered to leading order in v from the open string approximation for $p = 0, 4$. Equivalently, at leading order in small velocities, the phase shifts at *long* distances agree exactly, including coefficients, with the phase shifts at *short* distances. This means that the short distance, open string expansion can be used to reproduce the results of the large distance, closed string expansion. In the regime of agreement these calculations also agree with the eikonal approximation to supergravity. The realization that 0-branes are able to probe spacetime at very small distances while, in this manner, capturing low energy gravity at large distances, has led to the idea that 0-branes may have a particularly fundamental role in the final theory. The next section studies approaches by which we can try to extend the phase shift agreements between the open string approximation and supergravity to beyond the leading order in velocity.

4 0-brane Quantum Mechanics

In the preceding section the open string approximation to the kernels, Eq. 51 and Eq. 52, was found by explicitly expanding the exact one-loop expression Eq. 46. There is an alternative perspective on the truncation to open string modes that leads directly to the same expressions [8]. Here one considers the effective Lagrangian governing the low velocity interaction of n 0-branes:

$$\mathcal{L} = -\frac{1}{4} \text{Tr} F_{\alpha\beta} F^{\alpha\beta} + \bar{\lambda} \Gamma^\alpha D_\alpha \lambda \quad (53)$$

where

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] \quad (54)$$

$$D_\alpha \lambda = (\partial_\alpha + A_\alpha) \lambda \quad (55)$$

should be truncated to quantum mechanics (*i.e.* spatial derivatives omitted). The fields are in the fundamental representation of $SU(n)$. This Lagrangian represents, in a condensed form, all the amplitudes that can be derived using only the lightest open strings running between n 0-branes. For example, $0-0$ scattering with velocity v and impact parameter b can be described in this formalism by imposing the VEVs:

$$\langle X_1 \rangle = vt \frac{1}{2} \sigma^3 \quad (56)$$

$$\langle X_2 \rangle = b \frac{1}{2} \sigma^3 \quad (57)$$

on the $SU(2)$ fields $X_\alpha = 2\pi\alpha' A_\alpha$. The only subtlety that enters the calculation is that gauge fixing and the accompanying ghost terms must be considered carefully. (Some intermediate steps are written explicitly in [18]). The leading contributions to phase shifts derive from fluctuations and can be expressed as determinants arising from the quadratic terms in the Lagrangian. The heat kernel representation of these determinants reproduces the integral Eq. 46 with kernel Eq. 51. Specifically, the phase shift at large impact parameter derived from the 0-brane Lagrangian in Eq. 53 agrees at small velocities with the one derived from supergravity.

It is tremendously interesting that a theory based on such a simple Lagrangian, with no manifest appearance of familiar geometric concepts, nevertheless reproduces supergravity at large distances, although only at small velocities. A natural elaboration is to attempt also to account for the fully relativistic phase shift at long distances. One strategy is to augment the Lagrangian Eq. 53 with additional terms. This is natural because the role of velocity is played by the electric field in the world volume theory; so relativistic velocities correspond to strong fields where the effective interaction for 0-branes is the Born-Infeld Lagrangian, rather than its weak field limit in Eq. 53. Unfortunately, the complete supersymmetric non-abelian Born-Infeld Lagrangian is not known. However, any such Lagrangian must reduce to the known supersymmetrization of the Born-Infeld Lagrangian in the abelian limit [19] and should have a bosonic part that agrees with the recently derived non-abelian Born-Infeld Lagrangian that is valid up to certain ordering ambiguities that are yet to be understood [20]. For the present preliminary investigation we shall assume such a Lagrangian:

$$\mathcal{L} = -\sqrt{-\det M_{\alpha\beta}} = -\sqrt{-(M_{00} - M_{0i} M^{ij} M_{j0})} \sqrt{\det M_{ij}} \quad (58)$$

where

$$M_{\alpha\beta} = \eta_{\alpha\beta} + F_{\alpha\beta} - 2\bar{\lambda}\Gamma_\alpha D_\beta \lambda + \bar{\lambda}\Gamma^\gamma D_\alpha \lambda \bar{\lambda}\Gamma_\gamma D_\beta \lambda. \quad (59)$$

Here $\alpha, \beta = 0, \dots, 9$ and $i, j = 1, \dots, 9$. Assuming VEVs of the form Eqs. 56-57 the quadratic part of the Lagrangian becomes

$$\mathcal{L}_{\text{quad}} = -\sqrt{1-v^2} \left[1 + \frac{1}{1-v^2} \left[-\frac{1}{2} (F_{0i}^2 - \langle F_{0i} \rangle^2) + \frac{1}{2} v^2 F_{1i}^2 \right] + \frac{1}{4} F_{ij}^2 \right] \quad (60)$$

$$\frac{1}{1-v^2}[\bar{\lambda}(\Gamma_0 D_0 - v^2 \Gamma_1 D_1)\lambda + v\bar{\lambda}(\Gamma_0 D_1 - \Gamma_1 D_0)\lambda] - \bar{\lambda}\Gamma_i D_i \lambda] .$$

Taking a gauge fixed version of this quadratic action as a starting point, determinants can be evaluated as in the weak field calculation of [8] and a phase shift determined. It is:

$$\delta_{00} = \int_0^\infty \frac{dt}{2\pi t} e^{-\frac{tb^2}{2\pi\alpha'}} 4\pi \frac{(1 - \cos \frac{v}{1-v^2}t)^2}{\sin \frac{v}{1-v^2}t} \quad (61)$$

This phase shift, derived from a supersymmetrized nonabelian Born-Infeld action, is exactly the same as the integral Eq. 46 with the open-string, non-relativistic kernel Eq. 51 modified by the replacement:

$$v \rightarrow \frac{v}{1-v^2} \quad (62)$$

It is a check on our procedure and algebra that the integral Eq. 61 suffers no divergence at small t . At large impact parameters the phase-shift becomes:

$$2\delta_{00} \simeq \frac{1}{2\pi^3} \left(\frac{l_s}{b}\right)^6 \frac{1}{4} \left(\frac{v}{1-v^2}\right)^3 \quad (63)$$

Unfortunately, this does not reproduce the relativistic expression

$$2\delta_{00} \simeq \frac{1}{2\pi^3} \left(\frac{l_s}{b}\right)^6 \frac{(1 - \sqrt{1-v^2})^2}{v\sqrt{1-v^2}} \quad (64)$$

even though the proposed corrections indeed contribute terms that are leading in distance and subleading in velocity. Of course the non-abelian Born-Infeld action Eq. 58 is perhaps not entirely accurate and the result is therefore preliminary. On the other hand the correct relativistic generalization of the open string kernel (Eq. 51) must have a very restricted form for the integral Eq. 61 to be finite and it is difficult to imagine any prescription that would yield Eq. 64. Our procedure gives a very natural generalization of the open string kernel given in Eq. 51. However, this improvement is not sufficient to recover the full relativistic velocity dependence for $0-0$ scattering.

It should be emphasized that we do not know any systematic argument that an agreement with the relativistic results should be expected from our calculation that only uses an improved Lagrangian for the massless open string modes in the 0-brane theory. It is possible, for example, that the massive states of theory must be included. Our point is simply that the relativistic corrections from the open string point of view probably involve entirely different and more complicated physics than the non-relativistic treatment. This was already suspected of course, but the present calculation adds a new perspective.

5 M(atr)ix theory

Many of the string theory dualities derive from the relationship of the string theories in 10 dimensions to the conjectural M-theory in 11 dimensions that reduces to

supergravity at large scales. It has been recently suggested that M-theory can be defined non-perturbatively in the infinite momentum frame as the large n limit of the $SU(n)$ matrix model governing the low energy dynamics of a n 0-branes [11]. For the M(atrrix) proposal to succeed, it must reproduce the supergravity phase shifts for scattering of p-branes derived in this paper. The fact that the 0-brane quantum mechanics discussed in Sec. 4 reproduces supergravity results to leading order in velocity [8], coupled with the kinematics of the infinite momentum frame, has enabled the M(atrrix) model to successfully reproduce p-brane scattering to leading order in velocity [11, 21, 18]. In this section we show explicitly how this correspondence fails at higher order in velocity and discuss what it would take to improve the matching.

The kinematics of the M(atrrix) model imply that the 2-branes of the theory are bound to a large number of 0-branes. To study such systems we consider scattering of 0-branes from a bound state of 0-branes and 2-branes. This process can be analyzed in supergravity and in string theory. The phase-shifts at large distances are complicated functions of two variables that, as we shall show, agree in the two descriptions. For small transverse velocities and large boosts in the compact 11th dimension, M(atrrix) theory reproduces these results; indeed this is a remarkable success of the proposal. As we shall discuss it is not yet known how to calculate the full functions of velocity that we derive here purely within the framework of M(atrrix)-theory.

5.1 (2+0) Solutions in Supergravity

The Ramond-Ramond gauge field that couples to the 0-brane in 10 dimensions is simply the Kaluza-Klein component of the metric in the compactification of 11 dimensional supergravity on a circle. Therefore, from the point of view of supergravity, a bound state of a 2-brane with a collection of 0-branes can be constructed by boosting a 2 brane along the compact 11th dimension. Boosting the membrane in Eq. 8 gives the solution:

$$ds_{11}^2 = D_2^{-2/3}(-d\tilde{t}^2 + dx_1^2 + dx_2^2) + D_2^{1/3}(dx_3^2 + \cdots + d\tilde{x}_{11}^2) \quad (65)$$

$$C_3 = (D_2^{-1} - 1) d\tilde{t} \wedge dx_1 \wedge dx_2 \quad (66)$$

$$D_2 = 1 + \frac{P}{r^5} \equiv 1 + \frac{(Q/\cosh^2 \beta)}{r^5} \quad (67)$$

Here the coordinates \tilde{t} and \tilde{x}_{11} are the boosted ones:

$$\tilde{t} = t \cosh \beta + x_{11} \sinh \beta \quad (68)$$

$$\tilde{x}_{11} = t \sinh \beta + x_{11} \cosh \beta \quad (69)$$

and we have restored the 3-form gauge field that was not written explicitly in Eq. 8. When $\beta = 0$ the 2-brane solution (Eq. 8) is recovered. As $\beta \rightarrow \infty$ with $Q_0 = Q \tanh \beta \approx Q$ fixed, Eq. 65 reduces to the 0-brane solution lifted to 11 dimensions Eq. 21, except that the harmonic function $D_0 = 1 + \frac{q_0}{r^7}$ has been replaced by $D'_0 = 1 + \frac{Q_0}{r^5}$ due to the compactification of dimensions parallel to the 2-brane. The precise relationship between charge parameters is $Q_0 \omega_4 l_s^2 = q_0 \omega_6$ where ω_k is the volume of the

unit k-sphere (Eq. 44). This can be derived by requiring that the two configurations carry identical quantized charge (as defined in Sec. 2.7) or, equivalently, that their total charges agree when calculated using Gauss' law.

The solution is compactified to 10 dimensions using Eq.7 (for more details see also [22]). This yields 3-form and 1-form gauge fields:

$$C_{tx_1x_2} = \frac{\partial \tilde{t}}{\partial t} C_{\tilde{t}x_1x_2} = \cosh \beta \left(\frac{1}{D_2} - 1 \right) \xrightarrow{r \rightarrow \infty} -\frac{(Q/\cosh \beta)}{r^5} \quad (70)$$

$$A_t = -\frac{g_{11,t}}{g_{11,11}} = -\frac{(Q \tanh \beta)}{r^5} \left(1 + \frac{Q}{r^5} \right)^{-1} \xrightarrow{r \rightarrow \infty} -\frac{(Q \tanh \beta)}{r^5} \quad (71)$$

As in previous sections we define physical charges using the asymptotic behaviour of the potentials. With this convention the 2-brane charge is $q_2 = Q/\cosh \beta$ and the 0-brane charge is $Q_0 = Q \tanh \beta = q_2 \sinh \beta$. These formulae can be understood physically as follows². Due to the compactness of the 11th dimension a single 2-brane is described as a periodic array of 2-branes, each carrying a charge $q_2 = P$ as measured in the *boosted* frame $(\tilde{t}, \tilde{x}_{11})$. In the *stationary* frame (t, x_{11}) the spacing of the periodic array in the 11th dimension is Lorentz-contracted by a factor of $1/\gamma = 1/\cosh \beta$. The density of 2-branes is increased accordingly so that, after compactification, the apparent 2-brane charge becomes $q_2 = P \cosh \beta = Q/\cosh \beta$ as derived explicitly in Eq. 70. Similarly the 0-brane charge can be understood by writing it as $Q_0 = q_2 w/\sqrt{1-w^2}$ where the velocity w of the 2-brane in the 11th dimension is related to the boost through $\cosh \beta = \frac{1}{\sqrt{1-w^2}}$. Since the charge and mass of the 2-brane are equal we recognize the 0-brane charge as being simply equal to the relativistic momentum of the 2-brane in the 11th dimension.

5.2 0-(2+0) Scattering in Supergravity

Having obtained the (2+0) solution in 11 dimensions in Eq. 65 it is straightforward to apply the Hamilton-Jacobi method of Sec. 2.1 to analyze scattering of 0-branes from this background. The 0-brane trajectory is planar and the Hamilton-Jacobi equation becomes:

$$\begin{aligned} & -D_2 \left(\frac{\partial S}{\partial t} \right)^2 + D_2 \left(\frac{\partial S}{\partial x_{11}} \right)^2 + \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 \\ & - (D_2 - 1) \left(\frac{\partial S}{\partial t} \sinh \beta - \frac{\partial S}{\partial x_{11}} \cosh \beta \right)^2 = 0 \end{aligned} \quad (72)$$

Using the definitions of conserved quantities Eqs. 10-12 and the kinematical relations Eqs. 30-33, we find the eikonal approximation to the supergravity phase shifts as in previous sections:

$$\delta_{0,0+2} \approx \int_{r_{min}}^{\infty} \left(\frac{Q/\cosh^2 \beta}{2r^5} \right) \frac{(E \sinh \beta - p_{11} \cosh \beta)^2 + (E^2 - p_{11}^2)}{\sqrt{(E^2 - p_{11}^2) - J^2/r^2}} \quad (73)$$

²A similar discussion appears in [18] in the context of brane-antibrane scattering in M(atr)ix theory.

$$= \left(\frac{(Q/\cosh^2 \beta)}{2} \right) \frac{1}{b^4} \mu_0 I_2 \frac{v^2 + (\sinh \beta - \cosh \beta \sqrt{1-v^2})^2}{v\sqrt{1-v^2}} \quad (74)$$

Here v is the velocity of the probe 0-brane, b is the impact parameter and I_2 is defined in Eq. 36. Introducing a boost parameter $v = \tanh \pi\epsilon$ to describe the 10 dimensional velocity of the 0-brane, after a little algebra the phase shift for 0-(2+0) scattering can be compactly written as:

$$2\delta_{0,0+2} \approx \frac{\mu_0 I_2}{b^4} Q \frac{(\tanh \beta - \cosh \pi\epsilon)^2}{\sinh \pi\epsilon} \quad (75)$$

The construction of the (2+0) solution related the boost parameter β to the charge parameters as $\sinh \beta = \frac{Q_0}{q_2}$ and we also found that $Q_0 \omega_4 l_s^2 = q_0 \omega_6$. From these relations and the quantization conditions of Sec. 2.7 we find that the boost in the compact 11th dimension is quantized as $\sinh \beta = \frac{n_0}{n_2}$.

Putting all of this together we can rewrite Eq. 75 in terms of the number of 2-branes and 0-branes as:

$$2\delta_{0,0+2} \approx \sqrt{n_0^2 + n_2^2} \frac{1}{4\pi^2} \left(\frac{l_s}{b}\right)^4 \frac{\sqrt{1-v^2}}{v} \left(\frac{1}{\sqrt{1 + \left(\frac{n_2}{n_0}\right)^2}} - \frac{1}{\sqrt{1-v^2}} \right)^2 \quad (76)$$

This equation gives the phase shift for large impact parameter to all orders in velocity for scattering of a 0-brane off a bound state of n_0 0-branes and n_2 2-branes. In the 2-brane limit ($n_2 \rightarrow \infty$) and in the 0-brane limit ($n_0 \rightarrow \infty$), the phase shift reduces to the 0-2 and 0-0 results in Eq. 45. Note that, when inspecting Eq. 45 in the 0-0 case, the two compact dimensions parallel to the 2-brane must be averaged over, resulting in the replacement $(\frac{l_s}{b})^6 \rightarrow \frac{\pi}{2}(\frac{l_s}{b})^4$. An important point is that for general $\sinh \beta = \frac{n_0}{n_2}$ the phase shift Eq. 76 cannot be separated unambiguously into 0-brane and 2-brane contributions.

5.3 0-(2+0) Scattering in String Theory

We can compare the supergravity phase shift in Eq. 76 with the results from string theory where n_2 2-branes are bound to n_0 0-branes by turning on a magnetic flux on the 2-brane [23, 21, 18]. When the flux is $2\pi\alpha' F_{12} = \cot \pi\eta$ and the probe velocity is $v = \tanh(\pi\epsilon)$ the cylinder amplitude for 0-(2+0) scattering becomes [23, 18]:

$$2\delta_{0,0+2} = n_2 \int_0^\infty \frac{dt}{2\pi t} e^{-tb^2/2\pi\alpha'} (BJ)_{0,0+2} \quad (77)$$

where

$$\begin{aligned} (BJ)_{0,0+2} &= \frac{1}{2} \frac{\Theta'_1(0|it)}{\Theta_1(\epsilon t|it)} \left[- \left(\frac{f_2}{f_1} \right)^6 i \frac{\Theta_2(\eta t|it)}{\Theta_1(\eta t|it)} \frac{\Theta_2(\epsilon t|it)}{\Theta_2(0|it)} + \right. \\ &\quad \left. + \left(\frac{f_3}{f_1} \right)^6 i \frac{\Theta_3(\eta t|it)}{\Theta_1(\eta t|it)} \frac{\Theta_3(\epsilon t|it)}{\Theta_3(0|it)} - \left(\frac{f_4}{f_1} \right)^6 i \frac{\Theta_4(\eta t|it)}{\Theta_1(\eta t|it)} \frac{\Theta_4(\epsilon t|it)}{\Theta_4(0|it)} \right] \end{aligned} \quad (78)$$

For large impact parameter b it is appropriate to isolate contributions from the lightest closed strings as described in Sec. 3.1. This corresponds to expansion for small t which gives:

$$(BJ)_{0,0+2} \approx 2\pi t^2 \frac{(\cos \pi\eta - \cosh \pi\epsilon)^2}{\sinh \pi\epsilon \sin \pi\eta} \quad (79)$$

and the phase shift becomes:

$$2\delta_{0,0+2} \approx \frac{1}{4\pi^2} \left(\frac{l_s}{b}\right)^4 \frac{(\cos \pi\eta - \cosh \pi\epsilon)^2}{\sinh \pi\epsilon \sin \pi\eta} \quad (80)$$

The 0-brane charge induced by the flux is

$$n_0 = \frac{1}{2\pi} \int F = 2\pi\alpha' n_2 F_{12} = n_2 \cot \pi\eta \quad (81)$$

Here we have used the fact that in this paper the moduli were set to unity to avoid cumbersome notation; *i.e.* the lengths of the compact dimensions were chosen to be $l_s = 2\pi\sqrt{\alpha'}$ giving a volume of l_s^2 for the 2-torus on which the 2-brane is wrapped. In the supergravity calculations 0-brane charge arises from momentum in the 11th dimension and is parametrized by the boost β . Using Eq. 81 and the relation $\sinh \beta = n_0/n_2$ from the previous section we find that $\cot \pi\eta = \sinh \beta = \frac{n_0}{n_2}$. The physical interpretation is simply that the momentum density on the 2-brane is the same as the 0-brane charge density on the 2-brane. Trigonometric identities and the relation $\cosh \pi\epsilon = \frac{1}{\sqrt{1-v^2}}$ now let us rewrite the string theory phase shift as:

$$2\delta_{0,0+2} \approx \sqrt{n_0^2 + n_2^2} \frac{1}{4\pi^2} \left(\frac{l_s}{b}\right)^4 \frac{\sqrt{1-v^2}}{v} \left(\frac{1}{\sqrt{1 + (\frac{n_2}{n_0})^2}} - \frac{1}{\sqrt{1-v^2}}\right)^2 \quad (82)$$

This is exactly the same as the supergravity result Eq. 76. Previous comparisons between supergravity and string theory at large distances often assumed small impact velocity and large boost in the 11th dimension (*i.e.* $n_0 \gg n_2$).³ Here we find agreement between the full functional dependence on both these parameters.

5.4 0-(2+0) Scattering in M(atrix) Theory

The bound state between a 2-brane and large numbers of 0-branes is of particular interest because it is this composite object that appears in M(atrix)-theory where the fundamental objects - indeed the only objects - are 0-branes. In M(atrix) theory the calculation of phase shifts proceeds using the 0-brane quantum mechanics of Sec. 4. In this formalism an admixture of two-branes can be included by introducing fluxes

³After this paper was completed we became aware that the preprint [23] compares the full dependence on n_0 and n_2 , at leading order in velocity for the closely related process of 2-branes scattering off a (4+2) bound state. The same paper also compares the relativistic scattering of 0-branes from a (4+2) bound state and reports agreement between the velocity dependent potential computed from string theory and supergravity to all orders in velocity.

in the large n_0 0-brane theory. The precise correspondence between the fluxes and the charges is [11, 24]:

$$\frac{1}{2\pi\alpha'}[X_1, X_2] = i \tan \pi\eta \mathbf{I} = i \frac{n_2}{n_0} \mathbf{I} \quad (83)$$

The last equality is identical to Eq. 81; so the parameter η introduced here can be identified with the one that entered the string theory calculation. Note however that, because of the boosted kinematics in M(atrrix) theory, it is assumed *a priori* that $\tan \pi\eta \approx \pi\eta \ll 1$ and also that the transverse velocity of the probe 0-brane is $v = \tanh \pi\epsilon \simeq \pi\epsilon \ll 1$. The phase shift for scattering a 0-brane off this configuration of M(atrrix) theory is:

$$2\delta_{0,2+0} = n_2 \int \frac{dt}{2\pi t} e^{-\frac{b^2 t}{2\pi\alpha'}} 4\pi \frac{(\cos \pi\eta t - \cosh vt)^2}{(\sinh vt) (2 \sin \pi\eta t)} \quad (84)$$

In fact, this expression has only been derived for $n_2 = 1$ from the Matrix model [21, 18]. In that case it follows from the exact solution of a quantum mechanics problem involving numerous harmonic oscillators. The numerator derives from small shifts in the energy levels due to the background fields. The VEV in Eq. 57 that introduces the impact parameter acts as a mass term: hence the exponential damping with parameter b^2 . Moreover, the magnetic field (Eq. 83) also formally introduces a mass term which however is an operator that, after diagonalization and summation over the resulting “Landau” levels, gives the $2 \sin \pi\eta t$ in the denominator.

The scattering of a 0-brane off a bound state of 0-branes with fluxes can also be analyzed in string theory and is T-dual to the calculation described in the preceding section. In the string formalism Eq. 84 arises as an expansion keeping only the lightest open string modes of Eq. 78. In fact we actually arrived at Eq. 84 using this method. Treating configurations with $n_2 > 1$ directly within the Matrix model involves subtleties regarding the twisting of states of the 0-brane gauge theory with flux that we are in the process of studying.

Treating both η and v as quantities of order $1/n_0$ we expand Eq. 84 for large n_0 and find

$$2\delta_{0,0+2} \approx n_0 \frac{1}{4\pi^2} \left(\frac{l_s}{b}\right)^4 \frac{1}{4v} \left[\left(\frac{n_2}{n_0}\right)^2 + v^2\right]^2. \quad (85)$$

This agrees with Eq. 76, expanded for large n_0 ; so, to leading order in $1/n_0$ M(atrrix) theory reproduces the supergravity results at large distances. (Note again that the phase shifts have only been calculated directly for $n_2 = 1$ in the M(atrrix) approach.) In previous calculations the supergravity side of this equality was established by either boosting a potential inferred from the $0-2$ phase shift [21]⁴, or by expanding results from closed string theory [18]. In contrast we carried out the boost explicitly on the supergravity solution, checked that it agrees with the string theory result to all orders in velocity, and then proceeded to a direct and successful comparison between supergravity and M(atrrix) theory.

⁴In this approach a numerical discrepancy was reported.

The phase shift at leading order in the impact parameter can be isolated from Eq. 84 by expanding to leading order in t . It is evident that the resulting phase shift includes *only* the leading order term in velocity; so it certainly can not match the relativistic velocity dependence of the supergravity phase shift. Of course, the highly boosted kinematics of the M(atrix) model implies that terms that are subleading in velocity are suppressed by powers of $1/n_0$ so that the relativistic corrections are indeed small. On the other hand it might have been suspected that using the kinematic trick of first boosting to the infinite momentum frame, calculating nonrelativistic amplitudes, and then deboosting would permit a successful calculation of relativistic amplitudes in the stationary frame. Our calculation directly in the boosted frame shows that this is not possible.

The leading long distance supergravity phase shift in Eq. 76 depends in general on the probe velocity and two parameters - n_2 and n_0 , the number of 2-branes and the number of 0-branes in the bound state. The rather intricate dependence on these parameters reflects the kinematics of scattering in 11 dimensions. Recovering the complete functional form in the M(atrix) formalism, and not just the leading terms in v and $1/n_0$, will require control of $1/n_0$ corrections that has not been achieved at present. It is also possible that the M(atrix) Lagrangian will have to be improved, although the obvious improvement via a nonabelian Born-Infeld action does not seem to be sufficient, as shown in Section 4. It is clear that much more thought is necessary to fully recover the Lorentz invariant structure of supergravity from M(atrix) theory. Of course, this was already recognized for other reasons, but the scattering calculations of this paper add another perspective and provide a concrete functional form that should eventually be matched.

6 Conclusion

In this paper we have developed a technique for computing, within supergravity, the large impact parameter phase shifts for scattering of 0-branes from p-branes and bound states of p -branes. Our results are accurate to the leading order in impact parameter and to all orders in velocity. We have shown that phase shifts computed in string theory for large distance scattering of D-branes agree to all orders in velocity with our supergravity results. The phase shifts for 0-brane scattering can also be reproduced to leading order in velocity using the quantum mechanics governing a collection of 0-branes. Since this leading order matching has been so important in recent attempts to construct a non-perturbative definition of M-theory, we attempted to extend the matching to relativistic velocities. Our attempt involved a natural extension of the 0-brane effective Lagrangian to situations involving relativistic velocities - the nonabelian Born-Infeld action. Although this improved action contributed terms to the leading large distance phase shift that were subleading in velocity, they proved insufficient to restore agreement with supergravity. We then also compared the scattering of 0-branes from the 2-branes of $SU(n)$ M(atrix) theory to the analogous processes in supergravity. The kinematics of M(atrix) theory implies that the 2-branes of the theory are bound to a large number of 0-branes. Using the

techniques developed in this paper we computed, to all orders in velocity, the supergravity scattering of 0-branes from 2-branes bound to a large number of 0-branes. The expressions fail to match M(atrrix) theory results beyond the leading order in velocity. This shows that interactions of order $1/n$ will have to be accounted for more carefully in the M(atrrix) approach in order to successfully reproduce supergravity.

Acknowledgments: We would like to thank C. Bachas, J. Gauntlett, N. Manton, W. Taylor, A. Tseytlin, and especially C. Callan for helpful discussions and comments. After this paper was submitted to the archives we became aware of [23] where the complete relativistic velocity dependent potentials are calculated for scattering of 0-branes from a (4+2) bound state. F.L. would like to thank the Isaac Newton Institute for hospitality while part of this work was carried out. V.B. is supported in part by DOE grant DE-FG02-91-ER40671. F.L. is supported in part by DOE grant AC02-76-ERO-3071.

A Derivation of Charge Quantization

In this Appendix we turn to the derivation of Eq. 43. The standard reasoning is to identify the p -branes in supergravity with D -branes in string theory and follow world sheet considerations [16]. In the following we recover the same result using semiclassical methods in supergravity, using the method of [25]. The p -branes eqs. 3–6 are solutions to the equations of motion that follow from:

$$L = \frac{1}{16\pi G_N} \int d^{10}x \sqrt{-g} [e^{-2(\phi-\phi_\infty)} R + \frac{1}{2(p+2)!} F_{p+2}^2] \quad (86)$$

For $p = 0$ this Lagrangian follows from that of pure gravity in 11 dimensions, compactified to 10 dimensions using the Kaluza-Klein *ansatz* Eq. 7. By this reasoning it also follows from the periodicity of the 11th dimension that gauge functions satisfy the periodic identification $\Lambda \equiv \Lambda + g_{st} l_s$. Insisting that, after further compactification, the p -branes for various values of p are related by T -duality, the periodicity condition can be extended to all values of p . This simply assumes that the classical T -duality of the full Lagrangian can be extended to the semiclassical regime. Now, variation and integration by parts gives:

$$\delta L = \frac{1}{16\pi G_N} \int F^{rt\mu_1 \dots \mu_p} dS_8 \int dt \quad (87)$$

with no sum over repeated indices. The brane is invariant under gauge transformations so only the boundary term at infinity contributes. This is the crucial non-trivial property employed. As gauge variation we choose a pure gauge that depends only on time with $\Lambda_{\mu_1 \dots \mu_p}(\infty) - \Lambda_{\mu_1 \dots \mu_p}(-\infty) = g_{st} l_s$. The only non-zero component of the field strength is $F_{rt\mu_1 \dots \mu_p} = \partial_r D_p^{-1}$ so the charge is normalized as

$$\int F^{rt\mu_1 \dots \mu_p} dS_8 = q_p (7-p) \omega_{8-p} l_s^p = 2\pi q_p \omega_{6-p} l_s^p. \quad (88)$$

Signs are of no concern in these manipulations and the position of indices is irrelevant in flat space. Finally, using the relation between Newton's coupling constant and the string length $\frac{1}{16\pi G_N} = \frac{2\pi}{g_{\text{st}}^2 l_s^8}$ the variation becomes

$$\delta L = \frac{2\pi}{g_{\text{st}}^2 l_s^8} 2\pi q_p \omega_{6-p} l_s^p g_{\text{st}} l_s = \frac{(2\pi)^2 \omega_{6-p} q_p}{g_{\text{st}} l_s^{7-p}} . \quad (89)$$

The invariance of wave functions under gauge transformation quantizes this variation as $2\pi n_p$ and we find:

$$q_p = \frac{l_s^{7-p} g}{2\pi \omega_{6-p}} n_p \quad (90)$$

as stated in Eq. 43. In the derivation we included for simplicity no moduli but they could easily be restored. This convention corresponds to compactification on tori with the selfdual radius $R = \sqrt{\alpha'}$.

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